Elementary Theory of Some Novel

Hermite Polynomial Identities

An Exercise in Generatingfunctionology

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Introduction. The quantum theory of a pair of coupled oscillators recently brought to my attention—because the problem is so simple that it can be solved in two distinct ways—a population of identities, properties of the Hermite polynomials with which the quantum theory of oscillators is so replete, that appear in none of the standard handbooks and in no other literature of which I am presently aware. Notes written during the exploratory phase of this effort[1] manage to make the subject seem more obscurely/diffusely complex than in fact it is. Here I use (mainly) generating function methods standard to the general theory of orthogonal polynomials—and more particularly to the theory of Hermite polynomials—to obtain the novel identities by means that are relatively simple/transparent.

Simplest form of the identities in question.

$$\begin{aligned}
H_0\big(\tfrac{x-y}{\sqrt{2}}\big)H_1\big(\tfrac{x+y}{\sqrt{2}}\big) &= +\tfrac{1}{\sqrt{2}}H_0(x)H_1(y) + \tfrac{1}{\sqrt{2}}H_1(x)H_0(y)\\
H_1\big(\tfrac{x-y}{\sqrt{2}}\big)H_0\big(\tfrac{x+y}{\sqrt{2}}\big) &= -\tfrac{1}{\sqrt{2}}H_0(x)H_1(y) + \tfrac{1}{\sqrt{2}}H_1(x)H_0(y)
\end{aligned}$$

$$\begin{aligned}
H_0\big(\tfrac{x-y}{\sqrt{2}}\big)H_2\big(\tfrac{x+y}{\sqrt{2}}\big) &= +\tfrac{1}{2}H_0(x)H_2(y) + H_1(x)H_1(y) + \tfrac{1}{2}H_2(x)H_0(y)\\
H_1\big(\tfrac{x-y}{\sqrt{2}}\big)H_1\big(\tfrac{x+y}{\sqrt{2}}\big) &= -\tfrac{1}{2}H_0(x)H_2(y) \qquad\qquad\qquad\quad + \tfrac{1}{2}H_2(x)H_0(y)\\
H_2\big(\tfrac{x-y}{\sqrt{2}}\big)H_0\big(\tfrac{x+y}{\sqrt{2}}\big) &= +\tfrac{1}{2}H_0(x)H_2(y) - H_1(x)H_1(y) + \tfrac{1}{2}H_2(x)H_0(y)
\end{aligned}$$

[1] "Note concerning a possibly novel population of Hermite polynomial identites" and "Note concerning properties of the 'Hermite rotation matrices' latent in a novel population of Hermite polynomial identities," both dated March, 2020.

$$H_0\left(\tfrac{x-y}{\sqrt{2}}\right)H_3\left(\tfrac{x+y}{\sqrt{2}}\right)$$
$$= +\tfrac{1}{\sqrt{8}}H_0(x)H_3(y) + \tfrac{3}{\sqrt{8}}H_1(x)H_2(y) + \tfrac{3}{\sqrt{8}}H_2(x)H_1(y) + \tfrac{1}{\sqrt{8}}H_3(x)H_0(y)$$
$$H_1\left(\tfrac{x-y}{\sqrt{2}}\right)H_2\left(\tfrac{x+y}{\sqrt{2}}\right)$$
$$= -\tfrac{1}{\sqrt{8}}H_0(x)H_3(y) - \tfrac{1}{\sqrt{8}}H_1(x)H_2(y) + \tfrac{1}{\sqrt{8}}H_2(x)H_1(y) + \tfrac{1}{\sqrt{8}}H_3(x)H_0(y)$$
$$H_2\left(\tfrac{x-y}{\sqrt{2}}\right)H_1\left(\tfrac{x+y}{\sqrt{2}}\right)$$
$$= +\tfrac{1}{\sqrt{8}}H_0(x)H_3(y) - \tfrac{1}{\sqrt{8}}H_1(x)H_2(y) - \tfrac{1}{\sqrt{8}}H_2(x)H_1(y) + \tfrac{1}{\sqrt{8}}H_3(x)H_0(y)$$
$$H_3\left(\tfrac{x-y}{\sqrt{2}}\right)H_0\left(\tfrac{x+y}{\sqrt{2}}\right)$$
$$= -\tfrac{1}{\sqrt{8}}H_0(x)H_3(y) + \tfrac{3}{\sqrt{8}}H_1(x)H_2(y) - \tfrac{3}{\sqrt{8}}H_2(x)H_1(y) + \tfrac{1}{\sqrt{8}}H_3(x)H_0(y)$$

Here the $H_n(x)$ are the "physicists' Hermite polynomials" generated by

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \tfrac{1}{n!}H_n(x)t^n \tag{1}$$

of which the first few are

$$\begin{aligned}
H_0(x) &= 1\\
H_1(x) &= 2x\\
H_2(x) &= 4x^2 - 2\\
H_3(x) &= 8x^3 - 12x\\
H_4(x) &= 16x^4 - 48^2 + 12\\
H_5(x) &= 32x^5 - 160x^3 + 120x\\
H_6(x) &= 64x^6 - 480x^4 + 720x^2 - 120
\end{aligned}$$

$H_n(x)$ is a polynomial of degree n, even or odd according as n is.

Orthogonality. From

$$\int_{-\infty}^{+\infty} e^{-x^2}e^{2xs-s^2}e^{2xt-t^2}dx = \sqrt{\pi}\,e^{2st}$$

we have

$$\sum_m\sum_n\int_{-\infty}^{+\infty} e^{-x^2}\tfrac{1}{m!n!}H_m(x)H_n(x)s^mt^ndx = \sum_k \sqrt{\pi}\,\tfrac{1}{k!}2^ks^kt^k$$

whence

$$\int_{-\infty}^{+\infty} e^{-x^2}H_m(x)H_n(x)dx = 2^nn!\sqrt{\pi}\,\delta_{mn}$$

which establishes the sense in which the Hermite polynomials $H_n(x)$ are orthogonal. More convenient is some contexts are the normalized Hermite polynomials

$$h_n(x) = \frac{1}{\sqrt{2^nn!\sqrt{\pi}}}H_n(x) \tag{2}$$

$$\int_{-\infty}^{+\infty} e^{-x^2}h_m(x)h_n(x)dx = \delta_{mn} \tag{3}$$

Hermite bases. The real polynomials $p(x) = p_0x^0 + p_1x^1 + \cdots + p_nx^n$ of degree n collectively comprise a $(n+1)$-dimensional vector space $\mathcal{V}_n$, of which the polynomials $\{x^0, x^1, x^2, \ldots, x^n\}$ comprise the "natural basis" and the Hermite polynomials $\{H_0(x), H_1(x), H_2(x), \ldots, H_n(x)\}$ the "Hermite basis." If we endow $\mathcal{V}_n$ with the specific inner product structure

$$\langle p|q\rangle \equiv \int_{-\infty}^{+\infty} e^{-x^2} p(x)q(x)\,dx \tag{4}$$

then $\{h_0(x), h_1(x), h_2(x), \ldots, h_n(x)\}$ comprise the "orthonormal Hermite basis" in terms of which an arbitrary $p(x) \in \mathcal{V}_n$ can be developed

$$p(x) = \sum_{k=0}^{n} p_k h_k(x) \quad \text{with} \quad p_k = \langle p(x)|h_k(x)\rangle \tag{5}$$

Then $\langle p|p\rangle = p_0^2 + p_1^2 + \cdots + p_n^2$, and if $\langle p|p\rangle = 1$ then the $\{p_k\}$ are in effect elements of a unit vector. If $\langle p|p\rangle = \langle q|q\rangle = 1$ and $\langle p|q\rangle = 0$ then $\{p_k\}$ and $\{q_k\}$ are in effect elements of an orthogonal pair of unit vectors. These simple facts will play important roles in what follows.

Mathematica (after having evaluated twenty-five integrals) reports that the h-developments of these elements $\{x^0, x^1, x^2, x^3, x^4\}$ of the natural basis can be described

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = \pi^{\frac{1}{4}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{3}{\sqrt{2}} & 0 & \frac{\sqrt{3}}{2} & 0 \\ \frac{3}{4} & 0 & \frac{\sqrt{3}}{2} & 0 & \frac{\sqrt{3}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} h_0(x) \\ h_1(x) \\ h_2(x) \\ h_3(x) \\ h_4(x) \end{pmatrix}$$

The 0s above the principal diagonal reflect the fact that $x^n \perp h_m(x) : m > n$, while those below the principal diagonal indicate that $x^n \perp h_m(x)$ when $\{m, n\}$ are of opposite parity. One could use such information to construct the h-development of an arbitrary polynomial $p(x)$, though it is usually simpler to proceed from (5).

Hermite addition formula. "Bivariate polynomials of degree n" are constructs of the form

$$p(x, y) = \sum_{i,j:i+j\leqslant n} p_{ij}x^i y^j$$

$H_n(x+y)$—consider the example

$$\begin{aligned} H_4(x+y) &= 16(x+y)^4 - 48(x+y) + 12 \\ &= 16x^4 + 64x^3y + 96x^2y^2 + 64xy^3 + 16y^4 \\ &\qquad\qquad - 48x^2 - 96xy - 48t^2 + 12 \end{aligned}$$

—is bivariate of degree n. So are the products $\{H_k(x)H_{n-k}(y) : k = 0, 1, \dots, n\}$; consider the example

$$\begin{aligned} H_2(x)H_2(y) &= (4x^2-2)(4y^2-2)\\ &= 16x^2y^2 - 8x^2 - 8y^2 + 4 \end{aligned}$$

The "Hermite addition formula" presents the former as a linear combination of (slight variants of) the latter, and anticipates the structure of the novel identities of present interest.

The specific linear combination in question can be obtained as follows:

$$\begin{aligned} \sum_{n=0} \tfrac{1}{n!} H_n(x+y)\,t^n &= e^{2(x+y)t-t^2}\\ &= \exp\big[2xt - \tfrac{1}{2}t^2\big]\exp\big[2yt - \tfrac{1}{2}t^2\big]\\ &= \exp\big[2(x\sqrt{2})(t/\sqrt{2}) - (t/\sqrt{2})^2\big]\\ &\qquad\qquad \exp\big[2(y\sqrt{2})(t/\sqrt{2}) - (t/\sqrt{2})^2\big]\\ &= \sum_{u,v} \tfrac{1}{u!v!} H_u(x\sqrt{2})H_v(y\sqrt{2})(t/\sqrt{2})^{u+v}\\ &= \sum_{n=0} \tfrac{1}{n!} \sum_{k=0}^{n} 2^{-n/2} \tfrac{n!}{k!(n-k)!} H_k(x\sqrt{2})H_{n-k}(y\sqrt{2})\,t^n \end{aligned}$$

Evidently

$$H_n(x+y) = 2^{-n/2}\sum_{k=0}^{n}\binom{n}{k} H_k(x\sqrt{2})H_{n-k}(y\sqrt{2}) \tag{6.1}$$

which is the *Hermite addition formula*. Written

$$H_n\big(\tfrac{x+y}{\sqrt{2}}\big) = 2^{-n/2}\sum_{k=0}^{n}\binom{n}{k} H_k(x)H_{n-k}(y) \tag{6.2}$$

it acquires a qualitative resemblance—no accident, as will emerge!—to the novel identities illustrated on pages 1 & 2, and the argument suggests how those might be obtained.

Derivation of the identities in question. We proceed from

$$\begin{aligned} &\sum_{p,q} \tfrac{1}{p!q!} H_p\big(\tfrac{x-y}{\sqrt{2}}\big) H_q\big(\tfrac{x+y}{\sqrt{2}}\big) s^p t^q\\ &\qquad = \exp\big[2\big(\tfrac{x-y}{\sqrt{2}}\big)s - s^2\big]\exp\big[2\big(\tfrac{x+y}{\sqrt{2}}\big)t - t^2\big]\\ &\qquad = \exp\big[2x\big(\tfrac{t+s}{\sqrt{2}}\big) + 2y\big(\tfrac{t-s}{\sqrt{2}}\big) - (s^2+t^2)\big] \end{aligned}$$

which by $(s^2+t^2) = \big(\tfrac{t+s}{\sqrt{2}}\big)^2 + \big(\tfrac{t-s}{\sqrt{2}}\big)^2$ becomes

$$= \sum_{u,v} \tfrac{1}{u!v!} H_u(x)H_v(y)\big(\tfrac{t+s}{\sqrt{2}}\big)^u\big(\tfrac{t-s}{\sqrt{2}}\big)^v \tag{7}$$

On the left, $H_k\big(\tfrac{x-y}{\sqrt{2}}\big)H_{n-k}\big(\tfrac{x+y}{\sqrt{2}}\big)$ is bivariate of degree n for $k = 0, 1, 2, \dots, n$.

On the right, $H_u(x)H_v(y)$ is bivariate of degree $n = u+v$. To illustrate the utility of (7) I look to the case $n = 2$; *i.e.*, to the equations that result from identifying the left/right coefficients of $\{s^0t^2, s^1t^1, s^2t^0\}$. From

$$\begin{aligned}
&\tfrac{1}{0!2!}H_0\big(\tfrac{x-y}{\sqrt{2}}\big)H_2\big(\tfrac{x+y}{\sqrt{2}}\big)s^0t^2\\
&+\tfrac{1}{1!1!}H_1\big(\tfrac{x-y}{\sqrt{2}}\big)H_1\big(\tfrac{x+y}{\sqrt{2}}\big)s^1t^1\\
&+\tfrac{1}{2!0!}H_2\big(\tfrac{x-y}{\sqrt{2}}\big)H_0\big(\tfrac{x+y}{\sqrt{2}}\big)s^2t^0 = \tfrac{1}{0!2!}H_0(x)H_2(y)\big(\tfrac{t+s}{\sqrt{2}}\big)^0\big(\tfrac{t-s}{\sqrt{2}}\big)^2\\
&\qquad\qquad + \tfrac{1}{1!1!}H_1(x)H_1(y)\big(\tfrac{t+s}{\sqrt{2}}\big)^1\big(\tfrac{t-s}{\sqrt{2}}\big)^1\\
&\qquad\qquad + \tfrac{1}{2!0!}H_2(x)H_0(y)\big(\tfrac{t+s}{\sqrt{2}}\big)^2\big(\tfrac{t-s}{\sqrt{2}}\big)^0
\end{aligned}$$

and

$$\begin{aligned}
\big(\tfrac{t+s}{\sqrt{2}}\big)^0\big(\tfrac{t-s}{\sqrt{2}}\big)^2 &= \tfrac{1}{2}s^0t^2 - s^1t^1 + \tfrac{1}{2}s^2t^0\\
\big(\tfrac{t+s}{\sqrt{2}}\big)^1\big(\tfrac{t-s}{\sqrt{2}}\big)^1 &= \tfrac{1}{2}s^0t^2 \qquad\quad - \tfrac{1}{2}s^2t^0\\
\big(\tfrac{t+s}{\sqrt{2}}\big)^2\big(\tfrac{t-s}{\sqrt{2}}\big)^0 &= \tfrac{1}{2}s^0t^2 + s^1t^1 + \tfrac{1}{2}s^2t^0
\end{aligned}$$

we obtain

$$\begin{aligned}
&\tfrac{1}{0!2!}H_0\big(\tfrac{x-y}{\sqrt{2}}\big)H_2\big(\tfrac{x+y}{\sqrt{2}}\big)\\
&\qquad = \tfrac{1}{2}\tfrac{1}{0!2!}H_0(x)H_2(y) + \tfrac{1}{2}\tfrac{1}{1!1!}H_1(x)H_1(y) + \tfrac{1}{2}\tfrac{1}{2!0!}H_2(x)H_0(y)\\
&\tfrac{1}{1!1!}H_1\big(\tfrac{x-y}{\sqrt{2}}\big)H_1\big(\tfrac{x+y}{\sqrt{2}}\big)\\
&\qquad = -\tfrac{1}{0!2!}H_0(x)H_2(y) + \tfrac{1}{2!0!}H_2(x)H_0(y)\\
&\tfrac{1}{2!1!}H_2\big(\tfrac{x-y}{\sqrt{2}}\big)H_0\big(\tfrac{x+y}{\sqrt{2}}\big)\\
&\qquad = \tfrac{1}{2}\tfrac{1}{0!2!}H_0(x)H_2(y) - \tfrac{1}{2}\tfrac{1}{1!1!}H_1(x)H_1(y) + \tfrac{1}{2}\tfrac{1}{2!0!}H_2(x)H_0(y)
\end{aligned}$$

which exactly reproduces the second (3-element) batch of the identities presented on page 1, identities originally obtained[1] by several alternative—and in all cases less efficient/transparent—lines of argument.

The preceding identities were assembled "by hand." To automate assembly of the identities of composite degree n we from multiply (7) by $n!$ and have

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}H_k\big(\tfrac{x-y}{\sqrt{2}}\big)H_{n-k}\big(\tfrac{x+y}{\sqrt{2}}\big)\cdot s^kt^{n-k}\\
&\qquad\qquad = 2^{-n/2}\sum_{p=0}^{n}\binom{n}{p}H_p(x)H_{n-p}(y)\cdot(t+s)^p(t-s)^{n-p}
\end{aligned}$$

and writing

$$(t+s)^p(t-s)^{n-p} = \sum_{k=0}^{n}A_{kp}(n)\cdot s^kt^{n-k} \tag{8}$$

obtain

$$\sum_{k=0}^{n}\binom{n}{k}H_k\big(\tfrac{x-y}{\sqrt{2}}\big)H_{n-k}\big(\tfrac{x+y}{\sqrt{2}}\big)\cdot s^k t^{n-k}$$
$$= 2^{-n/2}\sum_{k=0}^{m}\sum_{p=0}^{n}\binom{n}{p}H_p(x)H_{n-p}(y)A_{kp}(n)\cdot s^k t^{n-k}$$

whence

$$H_k\big(\tfrac{x-y}{\sqrt{2}}\big)H_{n-k}\big(\tfrac{x+y}{\sqrt{2}}\big) = \sum_{p=0}^{n}\frac{2^{-n/2}}{\binom{n}{k}}\binom{n}{p}A_{kp}(n)\cdot H_p(x)H_{n-p}(y)$$
$$= \sum_{p=0}^{n}Q_{kp}(n)\cdot H_p(x)H_{n-p}(y) \tag{9.1}$$
$$Q_{kp}(n) = \frac{2^{-n/2}}{\binom{n}{k}}\binom{n}{p}A_{kp}(n) \tag{9.2}$$

From $H_0(\bullet) = 1$ and $\binom{n}{0} = 1$ we in the case $k = 0$ recover (6.2), which shows the identities (9) to be generalizations of the Hermite addition formulae.

Evaluation of the coefficients $A_{kp}(n)$ introduced at (8)—whence also of $Q_{kp}(n)$—is a tedious job best consigned to *Mathematica*.[2] In the case $n = 3$ we compute

$$\mathbb{Q}(n) \equiv \begin{pmatrix} Q_{00} & Q_{01} & Q_{02} & Q_{03}\\ Q_{10} & Q_{11} & Q_{12} & Q_{13}\\ Q_{20} & Q_{21} & Q_{22} & Q_{23}\\ Q_{30} & Q_{31} & Q_{32} & Q_{33}\end{pmatrix} = \tfrac{1}{\sqrt{8}}\begin{pmatrix} 1 & 3 & 3 & 1\\ -1 & -1 & 1 & 1\\ 1 & -1 & -1 & 1\\ -1 & 3 & -3 & 1\end{pmatrix} \tag{10}$$

which brings (9.1) into precise agreement with the identities that appear at the top of page 2.

Informative reformulation. From the orthogonality of the Hermite polynomials $H_m(x)$ follows that of the bivariate polynomials $H_{mn}(x,y) = H_m(x)H_n(y)$. Similarly, from the orthonormality of the normalized Hermite polynomials $h_m(x)$ follows that of the bivariate polynomials $h_{mn}(x,y) = h_m(x)h_n(y)$:

$$\langle h_{mn}|h_{pq}\rangle \equiv \iint_{-\infty}^{+\infty} e^{-x^2-y^2}h_{mn}(x,y)h_{pq}(x,y)\,dxdy = \delta_{mp}\delta_{nq} \equiv \delta_{mn,pq}$$

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[2] f[$n\_, p\_$]:=Expand[$(t+s)^p(t-s)^{n-p}$]
F[$n\_$]:=Table[f[n,p],{$p,0,n$}]
F[$n\_, p\_$]:=Table[Coefficient[F[n][[$p+1$]], $s^k t^{n-k}$],{$k,0,n$}]
A[$k\_, p\_, n\_$]:=F[n,p][[$k+1$]]
Q[$k\_, p\_, n\_$]:=$2^{-n/2}$Binomial[n,k]$^{-1}$ Binomial[n,p]A[k,p,n]

The functions $h_{k,n-k}(x,y) : k = 0,1,2,\ldots,n$ provide an orthonormal basis in the space of polynomials $p(x,y)$ of bivariate degree n.

Look now to the bivariate polynomials

$$\begin{aligned} g_{mn}(x,y) &\equiv\ h_m\big(\tfrac{x-y}{\sqrt{2}}\big)\,h_n\big(\tfrac{x+y}{\sqrt{2}}\big) \\ G_{mn}(x,y) &\equiv H_m\big(\tfrac{x-y}{\sqrt{2}}\big)H_n\big(\tfrac{x+y}{\sqrt{2}}\big) \end{aligned} \tag{11}$$

Into

$$\langle g_{mn}|g_{pq}\rangle = \iint_{-\infty}^{+\infty} e^{-x^2-y^2} h_m\big(\tfrac{x-y}{\sqrt{2}}\big)h_p\big(\tfrac{x-y}{\sqrt{2}}\big)h_n\big(\tfrac{x+y}{\sqrt{2}}\big)h_q\big(\tfrac{x+y}{\sqrt{2}}\big)\,dxdy$$

introduce new variables $X = \frac{x-y}{\sqrt{2}}$, $Y = \frac{x+y}{\sqrt{2}}$ and by

$$x^2+y^2 = X^2+Y^2, \qquad \det\begin{pmatrix} \partial x/\partial X & \partial x/\partial Y \\ \partial y/\partial X & \partial y/\partial Y \end{pmatrix} = 1$$

obtain

$$\langle g_{mn}|g_{pq}\rangle = \iint_{-\infty}^{+\infty} e^{-X^2-Y^2} h_m(X)h_p(X)h_n(Y)h_q(Y)\,dXdY = \delta_{mn,pq}$$

So the functions $G_{k,n-k}(x,y)$ provide an alternative orthogonal basis—and the functions $g_{k,n-k}(x,y) : k = 0,1,2,\ldots,n$ an alternative orthonormal basis—in the space of polynomials $p(x,y)$ of bivariate degree n.

The generalized addition formulae (9.1) can be formulated

$$\begin{pmatrix} G_{0,n} \\ G_{1,n-1} \\ G_{2,n-2} \\ \vdots \\ G_{n,0} \end{pmatrix} = \begin{pmatrix} Q_{0,0} & Q_{0,1} & Q_{0,2} & \cdots & Q_{0,n} \\ Q_{1,0} & Q_{1,1} & Q_{1,2} & \cdots & Q_{1,n} \\ Q_{2,0} & Q_{2,1} & Q_{2,2} & \cdots & Q_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_{n,0} & Q_{n,1} & Q_{n,2} & \cdots & Q_{n,n} \end{pmatrix} \begin{pmatrix} H_{0,n} \\ H_{1,n-1} \\ H_{2,n-2} \\ \vdots \\ H_{n,0} \end{pmatrix}$$

$$\boldsymbol{G}_n = \mathbb{Q}_n \boldsymbol{H}_n$$

But from (2) we have $H_n(x) = \sqrt{2^n n!\sqrt{\pi}}\,h_n(x)$ whence

$$H_{k,n-k} = 2^{n/2}\sqrt{n!\pi}\cdot\tbinom{n}{k}^{-\frac{1}{2}}h_{k,n-k}$$

$$\begin{pmatrix} H_{0,n} \\ H_{1,n-1} \\ H_{2,n-2} \\ \vdots \\ H_{n,0} \end{pmatrix} = 2^{n/2}\sqrt{n!\pi} \begin{pmatrix} \binom{n}{0}^{-\frac{1}{2}} & 0 & 0 & \cdots & 0 \\ 0 & \binom{n}{1}^{-\frac{1}{2}} & 0 & \cdots & \\ 0 & 0 & \binom{n}{2}^{-\frac{1}{2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n}{n}^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} h_{0,n} \\ h_{1,n-1} \\ h_{2,n-2} \\ \vdots \\ h_{n,0} \end{pmatrix}$$

$$\boldsymbol{H}_n = 2^{n/2}\sqrt{n!\pi}\cdot\mathbb{D}_n\boldsymbol{h}_n$$

Similarly $\boldsymbol{G}_n = 2^{n/2}\sqrt{n!\pi}\cdot\mathbb{D}_n\boldsymbol{g}_n$

So the generalized addition formulae, when expressed in terms of normalized Hermite polynomials, read

$$\boldsymbol{g}_n = \mathbb{R}_n \boldsymbol{h}_n \quad \text{with} \quad \mathbb{R}_n = \mathbb{D}_n^{-1}\mathbb{Q}_n\mathbb{D}_n \tag{12}$$

In the case $n = 3$ we from (10) obtain

$$\mathbb{R}_3 = \tfrac{1}{\sqrt{8}}\begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 1 \\ -\sqrt{3} & -1 & 1 & \sqrt{3} \\ \sqrt{3} & -1 & -1 & \sqrt{3} \\ -1 & \sqrt{3} & -\sqrt{3} & 1 \end{pmatrix} \tag{13}$$

The matrices $\mathbb{R}_n$, since they are seen in (12) to describe the linear relationship of one orthonormal basis to another, must be rotation matrices[3]: $\mathbb{R}_n\mathbb{R}_n^{\mathsf{T}} = \mathbb{I}$, and indeed, *Mathematica* reports that $\mathbb{R}_3$ is rotational and proper (unimodular):

$$\mathbb{R}_3\mathbb{R}_3^{\mathsf{T}} = \mathbb{I}, \qquad \det \mathbb{R}_3 = 1$$

The observaton[3] that the elements of $\mathbb{R}_n$ are inner products

$$\mathbb{R}_n = \begin{pmatrix} (h_{0,n-0}|g_{0,n-0}) & (h_{1,n-1}|g_{0,n-0}) & \cdots & (h_{n,n-n}|g_{0,n-0}) \\ (h_{0,n-0}|g_{1,n-1}) & (h_{1,n-1}|g_{1,n-1}) & \cdots & (h_{n,n-n}|g_{1,n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ (h_{0,n-0}|g_{n,n-n}) & (h_{1,n-1}|g_{n,n-n}) & \cdots & (h_{n,n-n}|g_{n,n-n}) \end{pmatrix}$$

provides an alternative (less efficient) way to construct such matrices; namely, by evaluating $(n+1)^2$ double integrals. By this means, *Mathematica* constructs $\mathbb{R}_3$ in about thirty seconds, and the unimodular rotation matrix

$$\mathbb{R}_4 = \tfrac{1}{4}\begin{pmatrix} 1 & 2 & \sqrt{6} & 2 & 1 \\ -2 & -2 & 0 & 2 & 2 \\ \sqrt{6} & 0 & -2 & 0 & \sqrt{6} \\ -2 & 2 & 0 & -2 & 2 \\ 1 & -2 & \sqrt{6} & -2 & 1 \end{pmatrix}$$

in less than a minute. Again, the top row (uniquely: no minus signs) produces the h-formulation of a Hermite addition formula.

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[3] The argument runs as follows: Suppose bases $\{|h_i)\}$ and $\{|g_i)\}$ both to be orthonormal and complete: $(h_i|h_j) = \delta_{ij}$ and $\sum_k |h_k)(h_k| = \mathbb{I}$; similarly $\{|g_i)\}$. Then

$$|g_j) = \textstyle\sum_k |h_k)(h_k|g_j) = \sum_k R_{jk}|h_k) \quad \text{with} \quad R_{jk} \equiv (h_k|g_j)$$

gives

$$(g_i|g_j) = \textstyle\sum_k (g_i|h_k)(h_k|g_j) = \sum_k (h_k|g_j)(g_i|h_k) = \delta_{ji}$$
$$\mathbb{R}\,\mathbb{R}^{\mathsf{T}} = \mathbb{I}$$

More general left-side arguments. The arguments $\frac{x\pm y}{\sqrt{2}}$ that appear on the left side of the identities displayed on pages 1 & 2—leading instances of (9.1)—arise from

$$\mathbb{R}(\alpha)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\alpha - y\sin\alpha \\ x\sin\alpha + y\cos\alpha \end{pmatrix} \quad : \quad \mathbb{R}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

in the case $\alpha = \pi/4$. We look now to the identities that result when the value of α is unrestricted; *i.e.*, to the H-development of the polynomials

$$H_k(ax-by)H_{n-k}(bx+ay) \quad : \quad \det\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a^2+b^2 = 1$$

Arguing as we did on page 4, we have

$$\begin{aligned}\sum_{p,q} \tfrac{1}{p!q!} H_p(ax-by)H_q(bx+ay)s^p t^q \\ &= \exp\big[2(ax-by)s - s^2\big]\exp\big[2(bx-ay)t - t^2\big] \\ &= \exp\big[2x(bt+as) + 2y(at-bs) - (s^2+t^2)\big]\end{aligned}$$

which by $(s^2+t^2) = \big(\frac{bt+as}{\sqrt{a^2+b^2}}\big)^2 + \big(\frac{at-bs}{\sqrt{a^2+b^2}}\big)^2$ becomes, subject to the unimodularity condition $a^2+b^2=1$,

$$= \sum_{u,v} \tfrac{1}{u!v!} H_u(x)H_v(y)\big(bt+as\big)^u\big(at-bs\big)^v$$

From this if follows in particular (set $p=k$, $q=n-k$; $u=p$, $v=n-p$) that

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k} H_k(ax-by)H_{n-k}(bx+ay)\cdot s^k t^{n-k} \\ &= \sum_{p=0}^{n}\binom{n}{p} H_p(x)H_{n-p}(y)\cdot(bt+as)^p(at-bs)^{n-p}\end{aligned}$$

Writing

$$(bt+as)^p(at-bs)^{n-p} = \sum_{k=0}^{n} A_{kp}(n;a,b)\cdot s^k t^{n-k}$$

(where the previous routine[2] serves to construct the coefficients $A_{kp}(n;a,b)$) we have—*subject to the unimodularity condition* (since it was evoked in the argument)—

$$H_k(ax-by)H_{n-k}(bx+ay) = \sum_{p=0}^{n} Q_{kp}(n;a,b)H_p(x)H_{n-p}(y) \tag{14.1}$$

$$Q_{kp}(n;a,b) = \frac{\binom{n}{p}}{\binom{n}{k}} A_{kp}(n;a,b) \tag{14.2}$$

Calculation in the case $n = 3$ gives

$$\mathbb{Q}_3(a,b) = \begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ -a^2b & a^3-2ab^2 & 2a^2b-b^3 & ab^2 \\ ab^2 & b^3-2a^2b & a^3-2ab^2 & a^2b \\ -b^3 & 3ab^2 & -3a^2b & a^3 \end{pmatrix} \tag{15}$$

$$\det \mathbb{Q}_3(a,b) = (a^2+b^2)^6$$

which gives back (10) in the unimodular case $a = b = 1/\sqrt{2}$. We can orchestrate general unimodularity by setting $a = \cos\alpha$, $b = \sin\alpha$. While the structure of the elements of the resulting $\mathbb{Q}_3(\alpha)$ is obvious, we note that the central terms can be written

$$\begin{aligned} a^3 - 2ab^2 &= \tfrac{1}{4}(\cos\alpha + 3\cos 3\alpha) \\ b^3 - 2a^2b &= \tfrac{1}{4}(\sin\alpha - 3\sin 3\alpha) \end{aligned}$$

Such trigonometric complexity becomes increassingly commonplace when one looks to the elements of $\mathbb{Q}_n(\alpha)$ with increasing n.

From (15) we obtain

$$\begin{aligned} \mathbb{R}_3(a,b) &= \mathbb{D}_3^{-1}\,\mathbb{Q}_3(a,b)\,\mathbb{D}_3 \\ &= \begin{pmatrix} a^3 & \sqrt{3}a^2b & \sqrt{3}ab^2 & b^3 \\ -\sqrt{3}a^2b & a^3-2ab^2 & 2a^2b-b^3 & \sqrt{3}ab^2 \\ \sqrt{3}ab^2 & b^3-2a^2b & a^3-2ab^2 & \sqrt{3}a^2b \\ -b^3 & \sqrt{3}ab^2 & -\sqrt{3}a^2b & a^3 \end{pmatrix} \\ \det \mathbb{R}_3(a,b) &= (a^2+b^2)^6 \end{aligned}$$

which is seen by

$$\mathbb{R}_3(a,b)\,\mathbb{R}_3^{\mathsf{T}}(a,b) = (a^2+b^2)^6 \cdot \mathbb{I}_3$$

to be rotational in unimodular cases. The structure of the manifestly unimodular matrix $\mathbb{R}_3(\alpha)$ is again obvious. $\mathbb{R}_n(\alpha)$ can be constructed similarly.

Conclusion: Generalized Hermite addition formulae. Writing

$$\begin{aligned} G_{m,n}(\alpha) \equiv G_{m,n}(x,y;\alpha) &= H_m(x\cos\alpha - y\sin\alpha)H_n(x\sin\alpha + y\cos\alpha) \\ g_{m,n}(\alpha) \equiv g_{m,n}(x,y;\alpha) &= h_m(x\cos\alpha - y\sin\alpha)\; h_n(x\sin\alpha + y\cos\alpha) \end{aligned}$$

we have been led, in the case "rank"$= n$, to a set of $n+1$ identities that can be written

$$\begin{pmatrix} G_{0,n}(\alpha) \\ G_{1,n-1}(\alpha) \\ G_{2,n-2}(\alpha) \\ \vdots \\ G_{n,0}(\alpha) \end{pmatrix} = \mathbb{Q}_n(\alpha) \begin{pmatrix} H_{0,n} \\ H_{1,n-1} \\ H_{2,n-2} \\ \vdots \\ H_{n,0} \end{pmatrix}, \quad \begin{pmatrix} g_{0,n}(\alpha) \\ g_{1,n-1}(\alpha) \\ g_{2,n-2}(\alpha) \\ \vdots \\ g_{n,0}(\alpha) \end{pmatrix} = \mathbb{R}_n(\alpha) \begin{pmatrix} h_{0,n} \\ h_{1,n-1} \\ h_{2,n-2} \\ \vdots \\ h_{n,0} \end{pmatrix}$$

where $H_{m,n} = G_{m,n}(0)$, $h_{m,n} = h_{m,n}(0)$. In the latter cases, the $\mathbb{R}_n(\alpha)$ matrices are unmimodular rotational, describe the relationship between two orthonormal bases in the space of polynomials $p(x,y)$ of bivariate degree n.

In the cases $G_{0,n}(\frac{1}{4}\pi)$ we recover the familiar Hermite addition formulae.